Announcements

1) Candidate Talk, tomorrow
2–3, CB 2070

2) HW 3 up, due next Thursday

3) Book notation: I will write a sequence like $(a_n)_{n\in\mathbb{N}}$. The book omits "$n\in\mathbb{N}$".

Example 1 (a divergent sequence)

A sequence is divergent if it does not converge to any real number.

Example: $a_n = \dfrac{(-1)^n + 1}{2}$

$a_1 = 0$

$a_2 = 1$

$a_3 = 0$

$a_4 = 1$

$\vdots$

Proving the sequence is divergent.

Assume, by contradiction, that the sequence converges to L.

Then $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\left| \frac{(-1)^n + 1}{2} - L \right| < \varepsilon$

$\forall\ n \geq N$.

Case 1 $L = 1$. Choose $\varepsilon = \frac{1}{2}$. Then for any odd number $n \in \mathbb{N}$,

$a_n = 0$, so

$$|a_n - 1| = |1| = 1 > \frac{1}{2} = \varepsilon$$

for all odd $n \in \mathbb{N}$. Since odd numbers are infinite, this is a contradiction - there is no $N \in \mathbb{N}$ after which all the odd terms are within $1/2$ of $L = 1$. Therefore, $L \neq 1$.

Case 2. $L \neq 1$. In this case, $|L-1| > 0$.

Choose $\varepsilon = \frac{|L-1|}{2}$.

Can assume $L \geq 0$ since all terms in the sequence are non-negative.

Then for all even $n \in \mathbb{N}$, $a_n = 1$, so

$$|a_n - L| = |1 - L|$$
$$> \frac{|1-L|}{2} = \varepsilon,$$

for all even $n \in \mathbb{N}$.

This is again a contradiction since the even numbers are infinite. So L must be one, but we showed this was a contradiction as well

Therefore L does not exist!

Sequence diverges.

See the book for a discussion on how to prove convergence of a sequence (general rules), p. 41

Definition: (partition) A partition of a set X is a disjoint collection of subsets of X whose union is X

For now, we'll concentrate on $X = \mathbb{N}$.

Notation: (even / odd)

we can partition the natural numbers into even numbers and odd numbers.

Notation: can write an even number as $2k$ for $k \in \mathbb{N}$.

Can write an odd number as $2k+1$ for $k \in \mathbb{N}$.

So: $(a_{2k+1})_{k \in \mathbb{N}}$ means all odd terms in a sequence

Proposition: (partitions & convergence)

If $\mathbb{N} = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_m$

for some infinite sets

K_i, $1 \leq i \leq m$, $m \in \mathbb{N}$,

then if

$$(a_n)_{n \in K_i} \text{ converge}$$

to the same number, then

$(a_n)_{n \in \mathbb{N}}$ converges to

that number

Proof: choose $\varepsilon > 0$ and suppose $(a_n)_{n \in K_i}$ converge to the same value L for all $1 \leq i \leq m$.

Then for each i, $\exists\, N_i \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \text{ for all } n \in K_i,\ n \geq N_i.$$

Let $N = \max\{N_1, N_2, \ldots, N_m\}$.

Then for all $n \geq N$,

$\exists\ i,\ 1 \leq i \leq m,\ n \in K_i$.

(partition) so

$|a_n - L| < \varepsilon$ since

$n \geq N \geq N_i$, which

implies $(a_n)_{n \in \mathbb{N}}$ converges

to L $\square$

You should try to understand this proof with $\mathbb{N} = K_1 \sqcup K_2$,

$K_1 =$ odd numbers

$K_2 =$ even numbers.

Definition: (bounded sequence)

A sequence $(a_n)_{n \in \mathbb{N}}$ is **bounded** if $\exists$ real numbers x and y with

$$x \leq a_n \leq y$$

for <u>all</u> $n \in \mathbb{N}$.

Examples:

1) $a_n = \frac{1}{n!}$ is bounded above by $y=1$ and below by $x=0$. (convergent)

2) $a_n = \frac{(-1)^n+1}{2}$ is bounded above by $y=1$ and below by $x=0$ (divergent)

3) $a_n = n+1$ is bounded below by 2, but not bounded above, so not bounded and divergent.

Proposition. A convergent sequence is bounded.

proof: Suppose $(a_n)_{n \in \mathbb{N}}$ converges to L. Let $\varepsilon = 2$

Then $\exists\ N \in \mathbb{N}$ such that

$|a_n - L| < 2$ for all $n \geq N$

This means

$-2 < a_n - L < 2$

$$-2 < a_n - L < 2$$

add L to the inequality.

$$-2 + L < a_n < 2 + L$$

for all $n \geq N$.

$\{a_n : n < N\}$ is a finite set!

Let $y = \max\{2+L, a_1, a_2, \ldots, a_{n-1}\}$

$x = \min\{-2+L, a_1, a_2, \ldots, a_{n-1}\}$.

$\square$

Notation: If $(a_n)_{n \in \mathbb{N}}$ converges to L, we write

$$a_n \longrightarrow L$$ or

$$\lim_{n \to \infty} a_n = L.$$